Robust Ordinal Regression Approach to Multiple Criteria Decision Support

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References


Plan

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- Why traditional MCDM methods may confuse their users?
- Towards „easy” preference information
- Principle of the ordinal regression – quick reminder of UTA method
- Construction of necessary and possible rankings using a set of all compatible additive value functions – UTAGMS and GRIP methods
- Illustrative examples
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Problem statement – multiple criteria ranking

- Consider a finite set \( A \) of actions (alternatives, solutions, objects) evaluated by \( n \) criteria from a consistent family \( F=\{g_1,\ldots,g_n\} \); \( I=\{1,\ldots,n\} \)

- Rank the actions from the best to the worst according to Decision Maker’s preferences
Problem statement – multiple criteria ranking

- The only objective information coming out from the formulation of the MCDM problem is the dominance relation in set $A$.

- Action $x \in A$ is non-dominated (Pareto-optimal) if and only if there is no other action $y \in A$ such that $g_i(y) \geq g_i(x)$, $i \in \{1, \ldots, n\}$, and on at least one criterion $j \in \{1, \ldots, n\}$, $g_j(y) > g_j(x)$. 

![Diagram showing non-dominated actions](attachment:image.png)
Problem statement – multiple criteria ranking

- Dominance relation is too poor – it leaves many actions non-comparable

- One can “enrich” the dominance relation, using preference information elicited from the Decision Maker (DM)

- Preference information permits to build a preference model that aggregates the vector evaluations of actions

- The preference model induces a preference relation in set $A$, which makes the actions more comparable

- A proper exploitation of the preference relation in $A$ leads to a final recommendation in terms of ranking
Why traditional MCDM methods may confuse their users?

- Traditional MCDM methods require a rich and difficult preference information:
  - many intracriteria and intercriteria parameters: thresholds, weights, ...
  - complete set of pairwise comparisons of actions on each criterion
  - complete set of pairwise comparisons of criteria
  - ...

- They suppose the DM understands the logic of a particular aggregation model:
  - meaning of weights: substitution ratios or relative strengths
  - meaning of lotteries (ASSESS)
  - meaning of indifference, preference and veto thresholds (ELECTRE)
  - meaning of the ratio scale of the intensity of preference (AHP)
  - meaning of „neutral” and „good” levels on particular criteria (MACBETH)
  - ...

Towards „easy” preference information

- The traditional methods appear to be too demanding of cognitive effort of their users.

- This is why we advocate for methods requiring „easy” preference information.

- „Easy” means natural and even partial.
Towards „easy” preference information

- Psychologists confirm that DMs are more confident exercising their decisions than explaining them.
- The most natural is a holistic pairwise comparison of some actions relatively well known to the DM, i.e. reference actions.
Towards „easy” preference information

- Psychologists confirm that DMs are more confident exercising their decisions than explaining them.
- The most natural is a holistic pairwise comparison of some actions relatively well known to the DM, i.e. reference actions.
Towards „easy” preference information

**Question:** what is the consequence of using on the whole set $A$ this information transformed to a compatible preference model?

Preference information:

- $x \succeq y$
- $z \succeq w$
- $x \succeq w$
- $y \succeq v$
- $u \succeq t$
- $z \succeq u$
- $u \succeq z$

Apply the preference model on $A$
The ordinal regression paradigm

- Ordinal regression paradigm emphasizes the discovery of intentions as an interpretation of actions rather than as a priori position.

- It is thus concordant with "posterior rationality" principle by March (1978) and "learning from example" used in AI and knowledge discovery.

- Other aggregation models inferred in this way:
  - Fishburn (1967) – trade-off weights of a value function
  - Jacquet-Lagreze & Siskos (1982) – additive p-l value function (UTA)
  - Zopounidis & Doumpos (1999) - additive p-l value function (UTADIS)
  - Greco, Matarazzo & Słowiński (1999) – decision rules or trees (DRSA – Dominance-based Rough Set Approach)
  - ...

The preference information is given in the form of partial preorder on a subset of reference actions $A^R \subseteq A$.

Additive value (or utility) function on $A$: for each $a \in A$

$$U(a) = \sum_{i=1}^{n} u_i[g_i(a)]$$

where $u_i$ are non-decreasing marginal value functions.
Principle of the ordinal regression – the UTA method  
(Jacquet-Lagreze & Siskos 1982)

- Marginal value $u_i(a)$ of action $a \in A$ is approximated by linear interpolation
- The scale of $u_i$ is a conjoint interval scale whatever the scale of $g_i$

![Diagram of piecewise linear marginal utility function]

Figure 1: Piecewise linear marginal utility function
The marginal value functions (breakpoint variables) are estimated by solving the LP problem

\[
\begin{align*}
\text{Min} & \quad E^{UTA} = \sum_{a \in A^R} (\sigma^+(a) + \sigma^-(a)) \\
\text{subject to} & \quad U(a) + \sigma^+(a) - \sigma^-(a) \geq U(b) + \sigma^+(b) - \sigma^-(b) + \varepsilon \quad \leftrightarrow \quad a \succ b \quad \forall a, b \in A^R \\
& \quad U(a) + \sigma^+(a) - \sigma^-(a) = U(b) + \sigma^+(b) - \sigma^-(b) \quad \leftrightarrow \quad a \sim b \\
& \quad u_i(x_i^{j+1}) - u_i(x_i^j) \geq 0 \quad j = 0, \ldots, \gamma_i - 1; \quad \forall i \in I \\
& \quad \sum_{i=1}^{n} u_i(\beta_i) = 1 \\
& \quad u_i(\alpha_i) = 0 \quad \forall i \in I \\
& \quad u_i(x_i^j) \geq 0, \quad \sigma^+(a) \geq 0, \quad \sigma^-(a) \geq 0, \quad \forall a \in A^R, \quad \forall i \quad \text{and} \quad j
\end{align*}
\]

where \( \varepsilon \) is a small positive constant, and \( \sigma^+ \) and \( \sigma^- \) are auxiliary variables (errors of approximation)
If $E_{\text{UTA}}^* = 0$, then the polyhedron of feasible solutions for $u_i(x_i)$ is not empty and there exists at least one value function $U(a)$ compatible with the complete preorder on $A^R$.

If $E_{\text{UTA}}^* > 0$, then there is no value function $U(a)$ compatible with the complete preorder on $A^R$ – three possible moves:

- increasing the number of linear pieces $\gamma_i$ for $u_i$
- revision of the complete preorder on $A^R$
- post optimal search for the best function with respect to Kendall’s $\tau$ in the area $F \leq F^* + \eta$

**Principle of the ordinal regression – the UTA method**

*(Jacquet-Lagreze & Siskos 1982)*
Example of UTA

- Ranking of 6 means of transportation

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Preference attitude: „economical-1”

\[ U(x) = u_C(x) + u_T(x) + u_P(x) \]
$U(x) = u_C(x) + u_T(x) + u_P(x)$
Preference attitude: „hurry-1”

\[ U(x) = u_C(x) + u_T(x) + u_P(x) \]
Preference attitude: „hurry-2”

\[ U(x) = u_C(x) + u_T(x) + u_P(x) \]
Preference attitude: “hurry-3”

\[ U(x) = u_C(x) + u_T(x) + u_P(x) \]
One should use **all compatible preference models** on set $A$

- **Question**: what is the consequence of using **all** compatible preference models on set $A$?

Apply all compatible instances on $A$

Preference information

- $x \succeq y$
- $z \succeq w$
- $y \succeq v$
- $u \succeq t$
- $z \succeq u$
- $u \succeq z$

What rankings will result?
Two rankings result: necessary and possible

\begin{align*}
\text{includes necessary ranking} & \quad \text{and does not include the complement of necessary ranking} \\
\end{align*}
Two rankings result: necessary and possible – effect of additional preference information

In the absence of any preference information:
- necessary ranking boils down to weak dominance relation
- possible ranking is a complete relation

For complete pairwise comparisons (complete preorder in $A$):
- necessary ranking $=$ possible ranking
Preoccupation of robustness: preference elicitation with constructive learning

- Decision maker
- Preference information
- Preference model
- Robustness analysis
- Set of compatible preference model parameters

Necessary and possible results (ranking, sorting)
Notation

- $G_i$ – domain of criterion $g_i$ ($G_i$ is finite or countably infinite)
- $G = \prod_{i=1}^{n} G_i$ – evaluation space
- $x, y \in G$ – profiles of actions in evaluation space
- $\succeq$ – weak preference (outranking) relation on $G$: for each $x, y \in G$
  \[ x \succeq y \iff \text{“} x \text{ is at least as good as } y \text{”} \]
  \[ x \succ y \equiv [x \succeq y \text{ and not } y \succeq x] \iff \text{“} x \text{ is preferred to } y \text{”} \]
  \[ x \sim y \equiv [x \succeq y \text{ and } y \succeq x] \iff \text{“} x \text{ is indifferent to } y \text{”} \]
- $x_i, y_i \in G_i$ – evaluations on criterion $g_i$
- $G_i = [\alpha_i, \beta_i]$ – value set of criterion $g_i$, $\alpha_i \leq \beta_i$, $i = 1, \ldots, n$, where $\alpha_i$ and $\beta_i$ are the worst and the best (finite) evaluations, resp.
The UTA\textsuperscript{GMS} method \cite{Greco2004}

- The marginal value function $u_i(x_i)$

Characteristic points of marginal value functions are fixed on actual evaluations of actions from set $A$.\[y, v, w, z \in A^R\]
The UTA\textsuperscript{GMS} method (Greco, Mousseau & Słowiński 2004)

- The marginal value function $u_i(x_i)$

$\begin{align*}
\text{Marginal values in characteristic points are unknown}
\end{align*}$
The UTA\textsuperscript{GMS} method (Greco, Mousseau & Słowiński 2004)

- The marginal value function $u_i(x_i)$

In fact, they are intervals, because all compatible value functions are considered
The UTA\textsuperscript{GMS} method (Greco, Mousseau & Słowiński 2004)

- The marginal value function $u_i(x_i)$

\[ u_i(x_i) \]

\[ y, v, w, z \in A^R \]

The area of all compatible marginal value functions
The UTAGMS method (Greco, Mousseau & Słowiński 2004)

- The marginal value function $u_i(x_i)$

In the area, the marginal compatible value functions must be monotone
The UTAGMS method (Greco, Mousseau & Słowiński 2004)

- The marginal value function $u_i(x_i)$

This means that the ordinal regression should not seek for $n$ piecewise-linear marginal value functions, but for any compatible additive value function.
The UTAGMS method (Greco, Mousseau & Słowiński 2004)

- The preference information is a partial preorder on a subset of reference actions $A^R \subseteq A$

- $B^R \subseteq A^R \times A^R$ is the set of $m$ pairs of reference actions compared by the DM

- A value function is called compatible if it is able to restore all pairwise comparisons from $B^R$

- In result, one obtains two rankings on set $A$, such that for any pair of actions $(x,y) \in A$:
  - in a necessary ranking (partial preorder): $x$ is ranked at least as good as $y$ iff $U(x) \geq U(y)$ for all compatible value functions
  - in a possible ranking (strongly complete): $x$ is ranked at least as good as $y$ iff $U(x) \geq U(y)$ for at least one compatible value function
The \textbf{UTA}^{GMS} method (Greco, Mousseau & Słowiński 2004)

- Formally, a general additive compatible value function is an additive function \( U(x) = \sum_{i=1}^{n} u_i(x_i) \) satisfying the following constraints:

\[
\begin{align*}
U(a) &\geq U(b) + \varepsilon \iff a > b \\
U(a) &= U(b) \iff a \sim b \\
u_i[g_i(a_{\tau_i(j)})] - u_i[g_i(a_{\tau_i(j-1)})] &\geq 0, \quad \forall i \in I, \quad j = 2, \ldots, m \\
u_i[g_i(a_{\tau_i(j)})] &\geq 0, \quad u_i[g_i(a_{\tau_i(m)})] \leq u_i(\beta_i), \quad \forall i \in I \\
u_i(\alpha_i) &= 0, \quad \forall i \in I \\
\sum_{i \in I} u_i(\beta_i) &= 1
\end{align*}
\]

where \( \varepsilon \) is a small positive constant and \( \tau_i \) is a permutation of action indices:

\[
g_i(a_{\tau_i(1)}) \leq g_i(a_{\tau_i(2)}) \leq \cdots \leq g_i(a_{\tau_i(m)}), \quad \text{for all} \quad a_{\tau_i(j)} \in A^R
\]
The UTA\textsuperscript{GMS} method (Greco, Mousseau & Słowiński 2004)

- For any pair of actions \((x,y)\in A\), and for available preference information represented by \(B^R\), preference of \(x\) over \(y\) is determined by compatible value functions \(U\) verifying set \(E(x,y)\) of constraints:

\[
U(a) \geq U(b) + \varepsilon \iff a \succ b
\]
\[
U(a) = U(b) \iff a \sim b
\]
\[
u_i\left(g_i^j\right) - u_i\left(g_i^{j-1}\right) \geq 0, \quad \forall i \in I, \quad j = 1, \ldots, \omega + 1
\]
\[
u_i\left(g_i^0\right) = 0, \quad \forall i \in I
\]
\[
\sum_{i \in I} u_i\left(g_i^{\omega+1}\right) = 1
\]

- \(g_i^j\) are characteristic points of \(u_i\):

\[
g_i^0 = a_i, \quad g_i^j = g_i\left(a_{\pi_i(j)}\right), \quad \text{for} \quad j = 1, \ldots, \omega, \quad g_i^{\omega+1} = \beta_i,
\]

where permutation \(\pi_i\) of the indices of actions \(a_{\tau_i(j)} \in A^R \cup \{x, y\}\) is such that:

\[
g_i\left(a_{\tau_i(1)}\right) \leq g_i\left(a_{\tau_i(2)}\right) \leq \ldots \leq g_i\left(a_{\tau_i(\omega)}\right), \quad \omega = \left|A^R \cup \{x, y\}\right|
\]
The UTA^{GMS} method

- \(\succeq^N\) means necessary preference relation

- Given a pair of actions \(x, y \in A\)

\[
x \succeq^N y \iff d(x, y) \geq 0
\]

where

\[
d(x, y) = \min_{s.t. E(x, y)} \{U(x) - U(y)\}
\]

- \(d(x, y) \geq 0\) means that for all compatible value functions \(x\) is at least as good as \(y\)

- For \(x, y \in A^R\):

\[
x \succeq y \Rightarrow x \succeq^N y
\]
The UTA\textsuperscript{GMS} method

- \( \preceq \) means possible preference relation

- Given a pair of actions \( x, y \in A \)

\[
x \preceq y \iff D(x, y) \geq 0
\]

where

\[
D(x, y) = \max_{\text{s.t. } E(x, y)} \{ U(x) - U(y) \}
\]

- \( D(x, y) \geq 0 \) means that for at least one compatible value function \( x \) is at least as good as \( y \)

- For \( x, y \in A^R \):

\[
x \succ y \Rightarrow \text{not } y \preceq x
\]
The UTA\textsuperscript{GMS} method

- Some properties:
  - $x \succeq_N y \Rightarrow x \succeq_P y$
  - $\succeq_N$ is a partial preorder (i.e. $\succeq_N$ is reflexive and transitive)
  - $\succeq_P$ is strongly complete (i.e. for all $x,y \in A$, $x \succeq_P y$ or $y \succeq_P x$) and negatively transitive (i.e. for all $x,y,z \in A$, not $x \succeq_P y$ and not $y \succeq_P z \Rightarrow$ not $x \succeq_P z$), (in general, $\succeq_P$ is not transitive)
  - $d(x,y) = \text{Min}\{U(x)-U(y)\} = -\text{Max}\{-[U(x)-U(y)]\} = -\text{Max}\{U(y)-U(x)\} = -D(y,x)$
The UTA\textsuperscript{GMS} method

- Elaboration of the rankings:

  - for the necessary preference relation being a partial preorder
    ($\succ_{N}^{N}$ is supported by all compatible value functions)
    
    preference: $x \succ_{N}^{N} y$ if $x \succ_{N}^{N} y$ and not $y \succ_{N}^{N} x$
    
    indifference: $x \sim_{N}^{N} y$ if $x \succ_{N}^{N} y$ and $y \succ_{N}^{N} x$
    
    incomparability: $x \not\sim_{N}^{N} y$ if not $x \succ_{N}^{N} y$ and not $y \succ_{N}^{N} x$

  - for the possible preference relation being complete
    ($\succ_{P}$ is supported by at least one compatible value function)
    
    preference: $x \succ_{P}^{P} y$ if $x \succ_{P}^{P} y$ and not $y \succ_{P}^{P} x$
    
    indifference: $x \sim_{P}^{P} y$ if $x \succ_{P}^{P} y$ and $y \succ_{P}^{P} x$
Example of $\text{UTA}^{\text{GMS}}$

- Ranking of 6 means of transportation

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Partial preorder graph

necessary ranking

D(x,y)
**UTA^{GMS} : an illustrative example**

**Ranking problem:** 20 actions evaluated on 5 criteria

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**Evaluation matrix**

**Empty dominance relation !**
UTA$^{GMS}$: an illustrative example

First iteration
UTA\textsuperscript{GMS} : an illustrative example

Second iteration
UTA$^{GMS}$ : an illustrative example

Third iteration
GRIP – Generalized Regression with Intensities of Preference
(Figueira, Greco & Słowiński 2008)

- GRIP extends the UTA$^\text{GMS}$ method by adopting all features of UTA$^\text{GMS}$ and by taking into account additional preference information:
  - comprehensive comparisons of intensities of preference between some pairs of reference actions,
    e.g. “$x$ is preferred to $y$ at least as much as $w$ is preferred to $z$”
  - partial comparisons of intensities of preference between some pairs of reference actions on particular criteria,
    e.g. “$x$ is preferred to $y$ at least as much as $w$ is preferred to $z$, on criterion $g_i \in F$”

A value function $U : \mathcal{R} \rightarrow [0, 1]$ is called compatible if it satisfies the constraints corresponding to DM’s preference information:

\begin{itemize}
  \item[a)] $U(x) \geq U(y)$ \iff $x \succeq y$
  \item[b)] $U(x) > U(y)$ \iff $x \succ y$
  \item[c)] $U(x) = U(y)$ \iff $x \sim y$
  \item[d)] $U(x) - U(y) \geq U(w) - U(z)$ \iff $(x,y) \succeq^{*} (w,z)$
  \item[e)] $U(x) - U(y) > U(w) - U(z)$ \iff $(x,y) \succ^{*} (w,z)$
  \item[f)] $U(x) - U(y) = U(w) - U(z)$ \iff $(x,y) \sim^{*} (w,z)$
  \item[g)] $u_{i}(x) \geq u_{i}(y)$ \iff $x \succeq_{i}^{*} y$, \quad i \in I$
  \item[h)] $u_{i}(x) - u_{i}(y) \geq u_{i}(w) - u_{i}(z)$ \iff $(x,y) \succeq_{i}^{*} (w,z)$, \quad i \in I
  \item[i)] $u_{i}(x) - u_{i}(y) > u_{i}(w) - u_{i}(z)$ \iff $(x,y) \succ_{i}^{*} (w,z)$, \quad i \in I
  \item[j)] $u_{i}(x) - u_{i}(y) = u_{i}(w) - u_{i}(z)$ \iff $(x,y) \sim_{i}^{*} (w,z)$, \quad i \in I
\end{itemize}
Moreover, the following normalization constraints should also be taken into account:

\( k) \quad u_i(\alpha_i) = 0, \quad i \in I \)

\( l) \quad \sum_{i \in I} u_i(\beta_i) = 1 \)
If constraints $a) - l$) are satisfied, then partial preorders on $A^R$ can be extended on $A$ by identification of two weak preference relations $\succeq^N$ and $\succeq^P$, and two binary relations comparing intensity of preference $\succeq^{*N}$ and $\succeq^{*P}$:

- For all $x, y \in A$, a necessary weak preference relation, $x \succeq^N y$:
  
  \[ U(x) \geq U(y) \text{ for all compatible value functions } U, \text{ thus} \]

  \[ x \succeq^N y \iff \inf_(a)-(l) \{U(x) - U(y)\} \geq 0 \]

- For all $x, y \in A$, a possible weak preference relation, $x \succeq^P y$:

  \[ U(x) \geq U(y) \text{ for at least one compatible value function } U, \text{ thus} \]

  \[ x \succeq^P y \iff \inf_(a)-(l) \{U(y) - U(x)\} \leq 0 \]
for all $x, y, w, z \in A$, a necessary relation of preference intensity $(x, y) \succeq^N (w, z)$:

$$[U(x)-U(y)]-[U(w)-U(z)] \geq 0$$ for all compatible value functions $U$,

$(x, y) \succeq^N (w, z) \iff \inf \{[U(x)-U(y)]-[U(w)-U(z)]\} \geq 0$

for all $x, y, w, z \in A$, a possible relation of preference intensity $(x, y) \succeq^P (w, z)$:

$$[U(x)-U(y)]-[U(w)-U(z)] \geq 0$$ for at least one compatible value function,

$(x, y) \succeq^P (w, z) \iff \inf \{[U(w)-U(z)]-[U(x)-U(y)]\} \leq 0$
GRIP – new contraints of the ordinal regression LP problem
(Figueira, Greco & Słowiński 2008)

- for all \(x, y, w, z \in A\), a necessary relation of preference intensity on criterion \(i \in I\), \((x, y) \succeq_{i}^{*N} (w, z)\):

\[
[u_i(x) - u_i(y)] - [u_i(w) - u_i(z)] \geq 0 \text{ for all compatible value functions } U,
\]

\[
(x, y) \succeq_{i}^{*N} (w, z) \iff \inf \{[u_i(x) - u_i(y)] - [u_i(w) - u_i(z)]\} \geq 0
\]

- for all \(x, y, w, z \in A\), a possible relation of preference intensity on criterion \(i \in I\), \((x, y) \succeq_{i}^{*P} (w, z)\):

\[
[u_i(x) - u_i(y)] - [u_i(w) - u_i(z)] \geq 0 \text{ for at least one compatible value function,}
\]

\[
(x, y) \succeq_{i}^{*P} (w, z) \iff \inf \{[u_i(w) - u_i(z)] - [u_i(x) - u_i(y)]\} \leq 0
\]
GRIP – Generalized Regression with Intensities of Preference
(Figueira, Greco & Słowiński 2008)

- GRIP permits to make preference intensity dependent on the part of criterion scale in which a difference of performances takes place, e.g.

\[(17.000; 19.000) \preceq_{\text{price}} (27.000; 30.000)\]

- GRIP can handle other kinds of preference information, like local tradeoffs, e.g.:

  "the increase of \(g_i(x)=6\) by 3 is at least as attractive as the decrease of \(g_j(x)=8\) by 2",

  which is represented by the LP constraint:

  \[u_i(9) - u_i(6) \geq u_j(8) - u_j(6)\]
GRIP – the linear programming problem

- In order to verify the truth or falsity of necessary and possible weak preference relations \( \succeq^N, \succeq^P \) and \( \succeq^{*N}, \succeq^{*P}, \succeq^{i*N}, \succeq^{i*P} \), one can use \textbf{LP}

- **Strict inequalities**, such as \( b), e), i) \), are rewritten as:
  \[
  b') \quad U(x) \geq U(y) + \varepsilon \\
  e') \quad U(x) - U(y) \geq U(w) - U(z) + \varepsilon \\
  i') \quad u_i(x) - u_i(y) \geq u_i(w) - u_i(z) + \varepsilon
  \]

- The result of GRIP is **independent** of \( \varepsilon \), while in UTA and in UTA\textsuperscript{GMS} the result is **dependent** on the value of \( \varepsilon \)

- \( x \succeq^P y \iff \varepsilon^* > 0 \), where \( \varepsilon^* = \text{Max} \varepsilon \), subject to constraints \( a)-l) \), with \( b), e), i) \) written as \( b'), e'), i' \) and \( U(x) \geq U(y) \)

- \( x \succeq^N y \iff \varepsilon^* \leq 0 \), where \( \varepsilon^* = \text{Max} \varepsilon \), subject to constraints \( a)-l) \), with \( b), e), i) \) written as \( b'), e'), i' \) and \( U(y) \geq U(x) + \varepsilon \)
**GRIP** – fundamental properties of $\succeq^N$, $\succeq^P$, $\succeq^{*N}$, $\succeq^{*P}$, $\succeq^*_N$, $\succeq^*_P$

- **Some properties:**
  - $x \succeq^N y \Rightarrow x \succeq^P y$,
  - $(x,y) \succeq^* N (w,z) \Rightarrow (x,y) \succeq^* P (w,z)$,
  - $(x,y) \succeq^* N (w,z) \Rightarrow (x,y) \succeq^* P (w,z)$, $g_i \in F$,
  - $\succeq^N$, $\succeq^{*N}$ and $\succeq^*_N$, $i \in I$, are **partial preorders**
  - $\succeq^P$, $\succeq^{*P}$ and $\succeq^*_P$ are **strongly complete** and negatively transitive
  (in general, $\succeq^P \not\succeq^P$, $\succeq^{*P}$ and $\succeq^*_P$ are not transitive)
Comparison of GRIP with MACBETH (Bana e Costa & Vansnick 1994)

- GRIP is using holistic and partial preference information on some pairs of actions.
- MACBETH requires partial preference information on all pairs of actions.
- Both GRIP and MACBETH deal with qualitative judgements about partial intensity of preference (equivalence classes of relation $\succ_i^*$ correspond to qualitative judgements of MACBETH), but in GRIP it may not be complete.
- GRIP uses a general additive value function as preference model.
- MACBETH uses a weighted sum of marginal values, thus it needs weights to aggregate interval scales on particular criteria.
- GRIP works with all compatible value functions, while MACBETH builds a single interval scale for each criterion, and a single vector of weights, even if many such scales and weights would be compatible with preference information.
Comparison of GRIP with AHP (Saaty, 1980)

- The preference information in AHP:
  - all criteria are to be compared pairwise on a fixed numerical scale
  - all actions are to be compared pairwise w.r.t. each criterion on the same numerical scale (1 to 9)
- The scale is a ratio scale expressing intensity of preference of element $i$ over element $j$
- The values of the scale (1-9) are supposed to be ratios of weights $w_i/w_j$
- The (approximate) values of weights are provided by the principal eigenvector and used in a single additive value function (weighted sum)
- In GRIP, no scale of intensity of preference is imposed a priori – instead, the marginal value functions are just a numerical representation of the original qualitative-ordinal information
- In GRIP, the marginal value functions depend mainly on holistic and incomplete judgements; no pairwise comparison of criteria is required
**GRIP : an illustrative example**

### Ranking problem: 20 actions evaluated on 5 criteria

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>$s_5$</th>
<th>$s_6$</th>
<th>$s_7$</th>
<th>$s_8$</th>
<th>$s_9$</th>
<th>$s_{10}$</th>
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<th>$s_{16}$</th>
<th>$s_{17}$</th>
<th>$s_{18}$</th>
<th>$s_{19}$</th>
<th>$s_{20}$</th>
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</tr>
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<tr>
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</tr>
<tr>
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<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

**Evaluation matrix**

**Empty dominance relation !**
GRIP: an illustrative example

Third iteration
GRIP: an illustrative example

Fourth iteration, after addition of **intensity condition**: \((s_{10}, s_2) \succ (s_1, s_2)\)
The idea of the "most representative" value function

- The principle of the most representative value function approach is: "one for all, all for one":
  - **one for all**: one value function is representing all compatible value functions
  - **all for one**: all compatible value functions contribute to the definition of the most representative value function

- The idea is to select among all compatible value functions the "most discriminant" value function for consecutive actions in the necessary ranking, i.e.
  that value function which maximizes the difference of scores between actions related by preference in the necessary ranking

- To tie-breaking, one can wish to minimize the difference of scores between actions not related by preference in the necessary ranking
1) Determine the necessary and the possible preferences in the considered set of actions.

2) For all pairs of actions \((a,b)\), such that \(a\) is necessarily preferred to \(b\) \((a \succ^N b\) but not \(b \succ^N a)\), add the following constraints to the linear programming constraints of GRIP: \(U(a) \geq U(b) + \varepsilon\).

3) Maximize \(\varepsilon\)

4) Add the constraint \(\varepsilon = \varepsilon^*\), with \(\varepsilon^* = \text{Max} \varepsilon\) from point 3), to the linear programming constraints of point 2)

5) For all pairs of actions \((a,b)\), such that neither \(a\) is necessarily preferred to \(b\) nor \(b\) is necessarily preferred to \(a\) \((\text{not } a \succ^N b\) and not \(b \succ^N a)\), add the following constraints to the linear programming constraints of GRIP: \(U(a) - U(b) \leq \delta\) and \(U(b) - U(a) \leq \delta\).

6) Minimize \(\delta\)
Equivalent, one-stage procedure

1) Determine the necessary and the possible preferences in the considered set of actions.

2) For all pairs of actions \((a, b)\), such that \(a\) is necessarily preferred to \(b\) \((a \succeq^N b\) but not \(b \succeq^N a\)), add the following constraints to the linear programming constraints of GRIP: \(U(a) \geq U(b) + \varepsilon\).

3) For all pairs of actions \((a, b)\), such that neither \(a\) is necessarily preferred to \(b\) nor \(b\) is necessarily preferred to \(a\) \((\text{not } a \succeq^N b \text{ and not } b \succeq^N a)\), add the following constraints to the linear programming constraints of GRIP: \(U(a) - U(b) \leq \delta\) and \(U(b) - U(a) \leq \delta\).

4) Maximize \(M\varepsilon - \delta\), where \(M\) is a „big value”
### Ilustrative example – ranking of students

<table>
<thead>
<tr>
<th>Student</th>
<th>Mathematics</th>
<th>Physics</th>
<th>Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>medium</td>
<td>medium</td>
<td>good</td>
</tr>
<tr>
<td>S2</td>
<td>good</td>
<td>good</td>
<td>medium</td>
</tr>
<tr>
<td>S3</td>
<td>medium</td>
<td>good</td>
<td>medium</td>
</tr>
<tr>
<td>S4</td>
<td>medium</td>
<td>medium</td>
<td>medium</td>
</tr>
<tr>
<td>S5</td>
<td>good</td>
<td>good</td>
<td>bad</td>
</tr>
<tr>
<td>S6</td>
<td>medium</td>
<td>bad</td>
<td>good</td>
</tr>
<tr>
<td>S7</td>
<td>bad</td>
<td>medium</td>
<td>good</td>
</tr>
</tbody>
</table>
Information on preferences given by the DM

- **Pairwise comparisons of some students**
  - S2 > S1
  - S4 > S5
  - S5 > S6

- **Overall intensity of preferences**
  - (S5,S6) $\succ^* (S2,S1)$

- **Intensity of preference relative to a single criterion**
  - (Good,Medium) $\succ^*_{Mathematics} (Medium, Bad)$
  - or, equivalently, (S2,S1) $\succ^*_{Mathematics} (S6,S7)$
GRIP on *Decision Deck* platform: www.decision-deck.org

**Evaluation Scores:**
- **good = 3**
- **medium = 2**
- **bad = 1**
### Reference Alternative Pair Info

<table>
<thead>
<tr>
<th>Pair</th>
<th>s2</th>
<th>s3</th>
<th>s4</th>
<th>s5</th>
<th>s6</th>
<th>s7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s2, s1)</td>
<td>3.0 - 3.0</td>
<td>3.0 - 2.0</td>
<td>3.0 - 2.0</td>
<td>3.0 - 2.0</td>
<td>3.0 - 2.0</td>
<td>3.0 - 1.0</td>
</tr>
<tr>
<td>(s3, s2)</td>
<td>3.0 - 3.0</td>
<td>3.0 - 2.0</td>
<td>3.0 - 2.0</td>
<td>3.0 - 2.0</td>
<td>3.0 - 2.0</td>
<td>3.0 - 1.0</td>
</tr>
<tr>
<td>(s4, s1)</td>
<td>2.0 - 3.0</td>
<td>2.0 - 3.0</td>
<td>2.0 - 2.0</td>
<td>2.0 - 2.0</td>
<td>2.0 - 2.0</td>
<td>2.0 - 1.0</td>
</tr>
<tr>
<td>(s5, s4)</td>
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<td>2.0 - 3.0</td>
<td>2.0 - 2.0</td>
<td>2.0 - 2.0</td>
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</tr>
<tr>
<td>(s1, s3)</td>
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<td>2.0 - 3.0</td>
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<td>2.0 - 2.0</td>
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</tr>
<tr>
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<td>1.0 - 2.0</td>
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</tr>
<tr>
<td>(s6, s2)</td>
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<td>2.0 - 3.0</td>
<td>2.0 - 2.0</td>
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<td>1.0 - 2.0</td>
<td>1.0 - 2.0</td>
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</tr>
<tr>
<td>(s6, s7)</td>
<td>2.0 - 3.0</td>
<td>2.0 - 3.0</td>
<td>2.0 - 2.0</td>
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<td>Necessary Preference Relation</td>
<td>Necessary Ranking Graph</td>
<td>Representative Ranking</td>
<td>Marginal Utilities</td>
<td>Dominance Relation</td>
<td>Possible Preference Relation</td>
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<tr>
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<td>s1: True</td>
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<tr>
<td>s2: True</td>
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<tr>
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<td>s3</td>
<td>s4</td>
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<td>s6</td>
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</tr>
</tbody>
</table>
The diagram shows the ranking of alternatives using a representative utility function. The alternatives are labeled as $s_1, s_2, s_3, s_4, s_7, s_5, s_6$. The ranking values range from 0.5 to 0.9. The alternative $s_2$ has the highest ranking at 0.9, followed by $s_3$ and $s_1$ with ranks of 0.8. The alternatives $s_4, s_7, s_5, s_6$ have ranks of 0.7, 0.6, and 0.5 respectively.
If the DM would wave:

(Good, Medium) $\succeq^*_{\text{Mathematics}}$ (Medium, Bad)
If the DM would add: 
(S5) ≻ (S7)
Conclusions

- We presented a new approach to **multicriteria ranking of actions**, based on preference elicitation with constructive learning.

- In **GRIP**, preference information is given by the DM in terms of:
  - pairwise comparisons of some reference actions
  - partial and comprehensive comparisons of intensities of preference for some pairs of reference actions

- The preference information is used within a **robust regression** approach to build a complete set of compatible additive value functions.

- The approach is robust with respect to preference relations on $A$ and on $A \times A$: the necessary relations are true for all compatible value functions, and the possible relations are true for at least one compatible value function.

- The most representative value function is build on necessary & possible relations.

- We extended **GRIP** on:
  - Multicriteria ranking with multiple decision makers
  - Multicriteria sorting with single or multiple decision makers
  - Interactive multiobjective optimization
  - Handling of preference information with gradual credibility
THANK YOU